BLOCH'S CONJECTURE AND CHOW MOTIVES

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Let X be a connected smooth projective complex surface. J. Murre [7] constructed a decomposition of Chow motives for X, i.e. there exist mutually orthogonal idempotents $\pi_i \in \mathrm{CH}^2(X \times X)_{\mathbb{Q}}$ as correspondences for $0 \le i \le 4$ such that $\sum_i \pi_i$ is equal to the diagonal and the action of π_i on $H^j(X, \mathbb{Q})$ is the identity for $i = j$, and vanishes otherwise. The decomposition is not uniquely characterized by the above properties. See also [8]. The theory of Chow motives would be rather complicated if the following condition is not satisfied:

(0.1) An idempotent of $\mathrm{CH}^2(X \times X)_{\mathbb{Q}}$ is zero if so is its cohomology class.

Here we can also consider a stronger condition:

(0.2) $\mathrm{Ker}(\mathrm{CH}^2(X \times X)_{\mathbb{Q}} \to H^4(X \times X, \mathbb{Q})(2))$ is a nilpotent ideal.

This is related to Beilinson's conjectures [2]. If (0.2) is true, the uniqueness of the projectors modulo inner automorphisms can be proved due to Beilinson. See [7, 7.3].

Let $h^i(X)$ denote the 'image' of the projector π_i in the motivic sense. Then $h^2(X)$ carries the Albanese kernel and the Neron-Severi group as well as the transcendental cycles. This is compatible with a conjecture of S. Bloch [3] that the vanishing of the transcendental cycles (i.e. that of p_g) would be equivalent to:

(0.3) The Albanese map is injective.

Note that the noninjectivity of the Albanese map in the case $p_g \ne 0$ is a theorem of D. Mumford [6], and Bloch's conjecture is proved at least if X is not of general type [4]. See also [1], [13] for some more cases.

In [7] the second projector π_2 is actually defined as the difference between the diagonal and the sum of the other projectors π_i. In the case $p_g = 0$, we can construct explicitly a projector $\tilde{\pi}_2$ which is homologically equivalent to π_2 and is orthogonal to the other projectors π_i $(i \ne 2)$. Then we can prove Bloch's conjecture if $\tilde{\pi}_2$ coincides with π_2 (modulo rational equivalence). So the conjecture is reduced to (0.1). Actually, we can show

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0.4. Theorem. *In the case $p_g = 0$, the above three conditions (0.1–3) are all equivalent, and the cube of the ideal in (0.2) is zero if it is nilpotent.*

The proof of (0.3) $\Rightarrow$ (0.2) uses an argument similar to [3] together with the bijectivity of the cycle map in the divisor case. Combined with the above mentioned result of [4], it implies

0.5. Theorem. *If $p_g = 0$ and X is not of general type (or, if X is as in [1], [13]), then the cube of the ideal in (0.2) is zero.*

We can show that the square of the ideal in (0.2) does not vanish if the irregularity $q\,(= \dim \Gamma(X, \Omega_X^1))$ is nonzero.

In Sect. 1, we review Murre's decomposition of Chow motives, and prove (0.1) $\Rightarrow$ (0.3). In Sect. 2 we show (0.3) $\Rightarrow$ (0.2) using a variant of the construction of [3].

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1. Chow motives

1.1. *Correspondences.* For smooth proper complex algebraic varieties X, Y such that X is purely n-dimensional, we define the group of correspondences with rational coefficients by

$$C^i(X, Y)_{\mathbb{Q}} = \mathrm{CH}^{n+i}(X \times Y)_{\mathbb{Q}}.$$

For $\xi \in C^i(X, Y)_{\mathbb{Q}}$ and $\eta \in C^j(Y, Z)_{\mathbb{Q}}$, the composition is denoted by $\eta \circ \xi \in C^{i+j}(X, Z)_{\mathbb{Q}}$. For $\zeta \in \mathrm{CH}^i(X)_{\mathbb{Q}}$, let

$$\Gamma_\zeta \in C^i(pt, X)_{\mathbb{Q}}\,(= \mathrm{CH}^i(X)_{\mathbb{Q}}) \tag{1.1.1}$$

be the element defined by ζ. For a morphism $f : X \to Y$, we denote by Γ_f the graph of f which belongs to $C^0(Y, X)$. Sometimes we will use the notation

$$f^* = \Gamma_f, \quad f_* = {}^t\Gamma_f. \tag{1.1.2}$$

Assume X, Y connected. By Hodge theory together with the Künneth decomposition and the duality, we have a canonical isomorphism

$$\frac{\mathrm{CH}^1(X \times Y)_{\mathbb{Q}}}{pr_1^*\mathrm{CH}^1(X)_{\mathbb{Q}} + pr_2^*\mathrm{CH}^1(Y)_{\mathbb{Q}}} = \mathrm{Hom}_{\mathrm{HS}}(H^{2n-1}(X, \mathbb{Q})(n-1), H^1(Y, \mathbb{Q})), \tag{1.1.3}$$

where the right-hand side is the group of morphisms of Hodge structures. See also [7], [12]. Let $\xi \in \mathrm{CH}_0(X)_{\mathbb{Q}}, \xi' \in \mathrm{CH}_0(Y)_{\mathbb{Q}}$ with degree one. Then the left-hand side of (1.1.3) is isomorphic to

$$\{\Gamma \in C^{1-n}(X, Y)_{\mathbb{Q}} \,|\, \Gamma \circ \Gamma_\xi = 0 \text{ and } {}^t\Gamma_{\xi'} \circ \Gamma = 0\}. \tag{1.1.4}$$

1.2. *Murre's construction.* Let X be a connected smooth projective variety of dimension $n \geq 2$. We choose and fix an embedding of X into a projective space. Let l denote the multiplication by the hyperplane section class. Let C be the intersection of $n-1$ generic smooth hyperplane sections. (Note that $[C] \in \mathrm{CH}^{n-1}(X)$ is independent of the choice of C.) By (1.1.3–4) there exists uniquely $\Gamma \in \mathrm{CH}^1(X \times X)_{\mathbb{Q}}$ such that

$$\Gamma \circ \Gamma_\xi = 0, \quad {}^t\Gamma_{\xi'} \circ \Gamma = 0, \tag{1.2.1}$$

$$\Gamma_* \circ l^{n-1} = id \text{ on } H^1(X, \mathbb{Q}). \tag{1.2.2}$$

We have ${}^t\Gamma = \Gamma$ if $\xi = \xi'$. Note that (1.2.1) implies

$$\Gamma_* : H^{i+2n-2}(X, \mathbb{Q})(n-1) \to H^i(X, \mathbb{Q}) \text{ vanishes for } i \neq 1. \tag{1.2.3}$$

Let $i : C \to X$ denote the inclusion morphism. Let

$$\pi' = \Gamma \circ i_* \circ i^*,$$

where $i_* = {}^t\Gamma_i, i^* = \Gamma_i$ as in (1.1.2). Following [12] we define

$$\pi_0 = \Gamma_{[X]} \circ {}^t\Gamma_{\xi'}, \quad \pi_1 = \pi' \circ (1 - {}^t\pi'/2), \quad \pi_{2n-1} = {}^t\pi_1, \quad \pi_{2n} = {}^t\pi_0,$$

where 1 denotes the diagonal of X. Then

$$(\pi_i)_* | H^j(X, \mathbb{Q}) = \delta_{i,j} id \quad \text{for } i = 0, 1, 2n-1, 2n. \tag{1.2.4}$$

If $n = 2$, we define $\pi_2 = 1 - \sum_{i \neq 2} \pi_i$.

1.3. Theorem. (Murre [7]). $\pi_i \circ \pi_j = \delta_{i,j} \pi_i$ *for* $\{i, j\} \subset \{0, 1, 2n-1, 2n\}$.

Outline of proof. We recall here some arguments of the proof which will be needed in the proof of the main theorems. See [7], [12] for details.

We have $\Gamma \circ i_* \circ i^* \circ \Gamma = \Gamma$ by (1.1.3–4), and ${}^t\Gamma \circ \Gamma = 0$ in $\mathrm{CH}^{2-n}(X \times X)_{\mathbb{Q}}$, because it is cohomologically zero by (1.2.3). So we get

$$\pi'^2 = \pi', \quad {}^t\pi' \circ \pi' = 0. \tag{1.3.1}$$

Then we can verify

$$\pi_1^2 = \pi_1, \quad \pi_{2n-1} \circ \pi_1 = 0, \quad \pi_1 \circ \pi_{2n-1} = 0. \tag{1.3.2}$$

We have furthermore

$$\pi_0 \circ \pi' = \pi_{2n} \circ \pi' = \pi' \circ \pi_0 = \pi' \circ \pi_{2n} = 0 \tag{1.3.3}$$

Indeed, $\pi_0 \circ \pi' = 0$ by (1.2.1), and $\pi' \circ \pi_0 = \pi_{2n} \circ \pi' = 0$, because $\pi' \circ \Gamma_{[X]} \in C^0(pt, X)$ is cohomologically zero (and similarly for ${}^t\Gamma_{[X]} \circ \pi'$). Finally the vanishing of $\pi' \circ \pi_{2n}$ follows from $i_* \circ i^* \circ \Gamma_{\xi'} \in C^{2n-1}(pt, X) = 0$.

Then we can verify the remaining assertions.

Remark. The Albanese map $\mathrm{CH}_0(X)^0_{\mathbb{Q}} \to \mathrm{Alb}_X(\mathbb{C})_{\mathbb{Q}}$ induces an isomorphism

$$(\pi_{2n-1})_*\mathrm{CH}_0(X)^0_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Alb}_X(\mathbb{C})_{\mathbb{Q}}.$$

If $n = \dim X = 2$, $(\pi_2)_*\mathrm{CH}_0(X)^0_{\mathbb{Q}}$ coincides with the kernel of the Albanese map with $\mathbb{Q}$-coefficients. See [7, 7.1].

1.4. *Construction of* $\tilde{\pi}_2$. Assume $n = \dim X = 2$ and $p_g = 0$. Let C_i be irreducible curves on X such that the cohomology classes of $[C_i]$ form a basis of $H^2(X, \mathbb{Q})(1)$. Let $\tilde{C}_i$ be the normalization of C_i, and $\tilde{C}$ the disjoint union of $\tilde{C}_i$ with $\tilde{i} : \tilde{C} \to X$ the canonical morphism. Let $A = (A_{i,j})$ be the intersection matrix of the C_i (i.e. $A_{i,j} = C_i \bullet C_j$). Let $B = (B_{i,j})$ be the inverse of A.

Let $\Gamma_B \in C^{-1}(\tilde{C}, \tilde{C})_{\mathbb{Q}} = \mathrm{CH}^0(\tilde{C} \times \tilde{C})_{\mathbb{Q}}$ defined by the matrix B. Let

$$\tilde{\Gamma} = \tilde{i}_* \circ \Gamma_B \circ \tilde{i}^*.$$

Since the composition

$$\tilde{i}^* \circ \tilde{i}_* : H^0(\tilde{C}, \mathbb{Q}) \to H^2(\tilde{C}, \mathbb{Q})(1)$$

is given by the matrix A (using the projection formula), we see that

$$\tilde{\Gamma}_* | H^i(X, \mathbb{Q}) = \delta_{i,2} id. \tag{1.4.1}$$

(The assertion for $i \neq 2$ follows from the definition of $\tilde{\Gamma}$.) Note that ${}^t\tilde{\Gamma} = \tilde{\Gamma}$ because B is symmetric. We define

$$\tilde{\pi}_2 = (1 - \pi') \circ \tilde{\Gamma} \circ (1 - {}^t\pi')$$

Then the symmetry ${}^t\tilde{\pi}_2 = \tilde{\pi}_2$ is clear.

1.5. Proposition. *$\tilde{\pi}_2$ is an idempotent, and is orthogonal to π_i for $i \neq 2$.*

Proof. We have $\pi' \circ \tilde{\pi}_2 = 0$ by (1.3.1). Since

$$\Gamma_B \circ \tilde{i}^* \circ \Gamma \in C^{-2}(X, \tilde{C})_{\mathbb{Q}} = \mathrm{CH}^0(X \times \tilde{C})_{\mathbb{Q}}$$

is cohomologically zero by (1.2.3), we get

$$\tilde{\Gamma} \circ \pi' = 0, \quad \tilde{\pi}_2 \circ \pi' = 0, \tag{1.5.1}$$

using (1.3.1). Then we have ${}^t\pi' \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ {}^t\pi' = 0$ by transpose, and

$$\pi_i \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ \pi_i = 0 \quad \text{for } i = 1, 3. \tag{1.5.2}$$

For $i = 0, 4$, we have $\tilde{\Gamma} \circ \pi_0 = \tilde{\Gamma} \circ \pi_4 = 0$ by $\Gamma_B \circ \tilde{i}^* \circ \Gamma_{[X]} \in C^{-1}(pt, \tilde{C}) = 0$ and $\tilde{i}^* \circ \Gamma_{\xi'} \in C^2(pt, \tilde{C}) = 0$. Then (1.5.2) holds also for $i = 0, 4$ using (1.3.3), because ${}^t\tilde{\Gamma} = \tilde{\Gamma}$.

Finally we have

$$\tilde{\pi}_1^2 = \tilde{\pi}_2. \tag{1.5.3}$$

Indeed, $\tilde{\Gamma}$ is an idempotent, because $\Gamma_B \circ \tilde{i}^* \circ \tilde{i}_* \circ \Gamma_B \in \mathrm{CH}^0(\tilde{C} \times \tilde{C})_{\mathbb{Q}}$ coincides with Γ_B (cohomologically). Then (1.5.3) follows from (1.5.1) and (1.3.1).

1.6. *Proof of* (0.1) $\Rightarrow$ (0.3). Applying (0.1) to $1 - (\sum_{i \neq 2} \pi_i + \tilde{\pi}_2)$, we get $\tilde{\pi}_2 = \pi_2$ by (0.1). So it is enough to show $(\tilde{\pi}_2)_* \mathrm{CH}^2(X)_{\mathbb{Q}} = 0$ by [7, 7.1] (because the Albanese kernel is torsion free by [9]). See Remark after (1.3). By the definition of $\tilde{\pi}_2$, it is enough to show $\tilde{i}^* \mathrm{CH}^2(X)_{\mathbb{Q}} = 0$. But this is clear.

2. Cycle maps and correspondences

2.1. Let X be a smooth proper complex variety with the structure morphism $a_X : X \to pt$. Let $\mathbb{Q}(j)$ denote the Tate Hodge structure of type $(-j, -j)$ (see [5]) which is naturally identified with a mixed Hodge Module on pt (see [10, (4.2.12)]). Then we have a cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}(a_X^* \mathbb{Q}, a_X^* \mathbb{Q}(p)) = \mathrm{Ext}^{2p}(\mathbb{Q}, (a_X)_* a_X^* \mathbb{Q}(p)), \tag{2.1.1}$$

where $\mathbb{Q}$ means $\mathbb{Q}(0)$, and Ext is taken in the derived category of mixed Hodge Modules on X or pt. See (4.5.18) of loc. cit. (The last isomorphism of (2.1.1) follows from the adjoint relation.) The target is isomorphic to $\mathbb{Q}$-Deligne cohomology, and (2.1.1) is an isomorphism for $p = 1$ as well-known. (This cycle map coincides with Deligne's cycle map which uses local cohomology, and can also be obtained by using the theory of Bloch and Ogus as was done by Beilinson and Gillet. In particular, its restriction to homologically equivalent to zero cycles coincides with Griffiths' Abel-Jacobi map.)

Let X, Y be smooth proper complex varieties. Assume X is purely n-dimensional. Then the cycle map induces

$$\begin{aligned} cl : C^i(X, Y)_{\mathbb{Q}} &\to \mathrm{Ext}^{2n+2i}(\mathbb{Q}, (a_{X \times Y})_* a_{X \times Y}^* \mathbb{Q}(n+i)) \\ &= \mathrm{Ext}^{2i}((a_X)_* a_X^* \mathbb{Q}, (a_Y)_* a_Y^* \mathbb{Q}(\mathrm{i}))\ . \end{aligned} \tag{2.1.2}$$

See [11, II]. This is an isomorphism for $n + i = 1$. By (3.3) of loc. cit, (2.1.2) is compatible with the composition of correspondences.

2.2. Proposition. *Let X, S be connected smooth proper complex varieties, and $\Gamma \in \mathrm{CH}^p(S \times X)_{\mathbb{Q}}$. Assume Γ is homologically equivalent to zero, and the cycle map (2.1.1) for X and p is injective. Then $\Gamma = \Gamma_1 + \Gamma_2$ where Γ_1 is supported on $D \times X$ for a divisor D on S, and $\Gamma_2 = [S] \times \xi$ for $\xi \in \mathrm{CH}^p(X)_{\mathbb{Q}}$.*

Proof. Let $H^{2p-1}(X, \mathbb{Q})_S$ denote a constant variation of Hodge structure on S such that the fibers are $H^{2p-1}(X, \mathbb{Q})$. This is identified with the direct image of $\mathbb{Q}_{S \times X}$ by the first projection. Since the restriction of Γ to $\{s\} \times X$ is homologically equivalent to zero for any $s \in S$, Γ determines a normal function

$$e \in \mathrm{Ext}^1(\mathbb{Q}_S, H^{2p-1}(X, \mathbb{Q})_S(p)),$$

where Ext is taken in the derived category of mixed Hodge Modules (using [10, 3.27]). Note that e can be identified with a section of $S \times J^p(X) \to S$ (where $J^p(X)$ is Griffith' intermediate Jacobian), if we replace rational coefficients with integral coefficients.

By the adjoint relation for $a_S : S \to pt$, we have a short exact sequence

$$0 \to \mathrm{Ext}^1(\mathbb{Q}, H^{2p-1}(X,\mathbb{Q})(p)) \to \mathrm{Ext}^1(\mathbb{Q}_S, H^{2p-1}(X,\mathbb{Q})_S(p)) \\ \to \mathrm{Hom}(\mathbb{Q}, H^1(S,\mathbb{Q}) \otimes H^{2p-1}(X,\mathbb{Q})(p)) \to 0,$$

where the first Ext and the last Hom are taken in the category of mixed Hodge structures. Since Γ is homologically equivalent to zero, the image of e in the last term is zero, and hence e comes from the first term, i.e. it is constant. So there exists $\xi \in \mathrm{CH}^p(X)_{\mathbb{Q}}$ such that ξ is homologically equivalent to zero and e is the image of $[S] \times \xi$. Then replacing Γ with $\Gamma - [S] \times \xi$, we may assume $e = 0$.

Let k be a finitely generated subfield of $\mathbb{C}$ such that X, S and Γ are defined over k, i.e. there exist smooth proper k-varieties X_k, S_k and $\Gamma_k \in \mathrm{CH}^p(S_k \times_k X_k)_{\mathbb{Q}}$ with isomorphisms $X = X_k \otimes_k \mathbb{C}$, etc. Let $K = k(S_k)$ the function field of S_k, and X_K the generic fiber of the first projection of $S_k \times_k X_k$. Let $\Gamma_K \in \mathrm{CH}^p(X_K)_{\mathbb{Q}}$ denote the restriction of Γ_k. Then it is enough to show $\Gamma_K = 0$.

We choose an embedding $K \to \mathbb{C}$ extending $k \to \mathbb{C}$. Since $e = 0$, the image of $\Gamma_K \otimes_K \mathbb{C} \in \mathrm{CH}^p(X)_{\mathbb{Q}}$ by the cycle map is zero, because $\Gamma_K \otimes_K \mathbb{C}$ is identified with the restriction of Γ to $\{s\} \times X$ where $s \in S$ corresponds to the embedding $K \to \mathbb{C}$. So $\Gamma_K \otimes_K \mathbb{C}$ is zero by hypothesis. Then the assertion follows from the injectivity of $\mathrm{CH}^p(X_K)_{\mathbb{Q}} \to \mathrm{CH}^p(X)_{\mathbb{Q}}$.

2.3. Theorem. *Let X be a connected smooth proper complex surface such that the Albanese map for X is injective. Then the cube of the ideal in (0.2) is zero.*

Proof. Let $\Gamma \in \mathrm{CH}^2(X \times X)_{\mathbb{Q}}$ that is homologically equivalent to zero. Then $\Gamma = \Gamma_1 + \Gamma_2$ such that Γ_1 is supported on $D \times X$ and $\Gamma_2 = [X] \times \xi$ as in (2.2). We have to show

$$\Gamma''\circ\Gamma'\circ\Gamma_i = 0 \ \ (i = 1, 2) \tag{2.3.1}$$

for any $\Gamma', \Gamma'' \in C^0(X, X)$ which are homologically equivalent to zero.

For $i = 1$, let Y denote the normalization of D with $f : Y \to X$ the canonical morphism. Then there exists $\Gamma'_1 \in C^0(Y, X)_{\mathbb{Q}}$ such that $\Gamma_1 = \Gamma'_1\circ f^*$. By the injectivity of the cycle map in the divisor case, it is enough to show that the image of $\Gamma''\circ\Gamma'\circ\Gamma'_1$ by (2.1.2) is zero. Since Γ', Γ'' are homologically equivalent to zero, and Ext^2 vanishes, the assertion follows by using for example a (noncanonical) decomposition

$$(a_X)_* a_X^* \mathbb{Q} \simeq \oplus_i H^i(X, \mathbb{Q})[-i]$$

in the derived category of mixed Hodge Modules on pt (or equivalently, of graded-polarizable mixed Hodge structures). See [10, (4.5.4)].

The argument is similar for $i = 2$ by using the injectivity of the Albanese map, because it is enough to show that the image of $\Gamma''\circ\Gamma'\circ\Gamma_\xi \in C^2(pt, X)$ by the cycle map (2.1.2) is zero. This completes the proof of (2.3).

2.4. *Remark.* The square of the ideal in (0.2) is nonzero if $H^1(X,\mathbb{Q}) \ne 0$. Indeed, let C be a hyperplane section of X with the inclusion morphism $i : C \to X$. By Hodge theory we have a divisor D on $C \times X$ such that $D_* : H^i(C,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ is zero for $i \ne 1$ and $D_* \circ i^* : H^1(X,\mathbb{Q}) \to H^1(X,\mathbb{Q})$ is an isomorphism. Let k be a finitely generated subfield of $\mathbb{C}$ such that X, C, D are defined over k, i.e. there exist X_k, C_k, D_k with isomorphisms $X_k \otimes_k \mathbb{C} = X$, etc. Let $D_i = (pr_i \times id)^* D_k$ where $pr_i : C_k \times_k C_k \to C_k$ are the natural projections for $i = 1, 2$. Let $K = k(C_k \times_k C_k)$, and $\Gamma_{K,i}$ be the restriction of D_i to the generic fiber of $C_k \times_k C_k \times_k X_k \to C_k \times_k C_k$. Then $\Gamma_{K,1} \times_K \Gamma_{K,2}$ is the restriction of $D_k \times_k D_k$. Let $\Gamma_i = \Gamma_{K,i} \otimes_K \mathbb{C} \in C^1(pt, X)_{\mathbb{Q}}$. Then the composition of $\Gamma_{[C]} \circ {}^t\Gamma_1$ and $\Gamma_1 \circ {}^t\Gamma_{[C]}$ is equal to a nonzero multiple of $\Gamma_2 \circ {}^t\Gamma_1$, and this is nonzero.

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